RESEARCH ARTICLE

Pre-Local Function in Ideal Topological Spaces

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ABSTRACT: The aim of this paper is to introduce the notation of pre-local function $A^{p^*}(I, \tau)$ by using pre-open sets in an ideal topological space (X, τ, I). Some properties and characterizations of a pre-local function are explored Pre-compatible spaces are also defined and investigated. Moreover, by using $A^{p^*}(I, \tau)$ we introduce an operator ψ: $P(X)\rightarrow\tau$ satisfying $\psi(A) = X-(X - A)^{p^*}$ for each $A \in P(X)$ and we discuss some characterizations this operator by use pre-open sets.

KEYWORDS: Ideal topological space; Local-function; Pre-local function; Compatible space; Pre-compatible space; Pre-open setoperator ψ.

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1. INTRODUCTION

The concept of ideal topological spaces was first studied by Kuratowski in 1933 [1] and Vaidyanathaswamy in 1960 [2]. Jankovic and Hamlett in 1990 [3] was investigated further the properties of ideal topological spaces. Compatibility of the topology τ with an ideal I was first introduced by Njastad in 1966 [4]. Jankovic and Hamlett in 1990[3] have studied some properties and characteristics of compatible spaces. It is given in [3] that for an ideal I of introduced subsets of X, $\tau \sim I$ if and only if(X, τ) is hereditarily compact. In this paper we will introduce a pre-local function by using pre-open sets and deduce some results. We introduce and characterize pre-compatible spaces and then we introduce an operator $\psi : P(X)\rightarrow\tau$ satisfying $\psi(A) = X-(X-A)^*$for each $A \in P(X)$ and we discuss some characterizations this operator by using pre-open sets.

2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), Cl(A) and Int(A) denote the closure and interior of A in(X, τ), respectively. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

1) $A \in I$ and $B \subseteq A$ implies $B \in I$,
2) $A \in I$ and $B \in I$ implies $A\cup B \in I$.

An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subseteq X$, $A^*(I, \tau) = \{x \in X{:}A\cap U \notin I$ for every $U \in \tau(x)\}$ is called the local function of A with respect to I and τ, [3, 4]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by the base $\beta(I, \tau) = \{U - E{:}\ U \in \tau$ and $E\in I\}$. It is known in [3] that $\beta(I, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(I)$ is denoted by τ^*. Recall that A is saidto be $*$-dense in itself if $A \subseteq A^*$, A is τ^*-closed if and only if $A^*\subseteq A$ and A is$*$-perfect if $A = A^*$. For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will, respectively, denote the closure and interior of A in (X, τ^*).

Definition.2.1.[8] Let (X, τ) be a topological space. A subset A of X is said to be pre-open if $A \subseteq Int(Cl(A))$. The complement of a pre-open set is said to be pre-closed. The collection of all pre-open (resp. pre closed) sets in X is denoted by PO(X) (resp. PC(X)).

Remark.2.1. We denote all pre-open sets forms a topology τ^p.

Definition.2.2.[8] Let (X, τ) be a topological space. Then the pre-interior and the pre-closure of A in X defined as $PInt(A) = \cup\{U : U \subseteq A, U\in\tau^p\}$ and $PCl(A) = \cap\{F : A\subseteq F, X - F \in\tau^p\}$.From definition, PInt(A) is a pre-open set and PCl(A) is a pre-closed set.

3. Pre-local functions

Definition.3.1. Let (X, τ, I) be an ideal topological space. For a subset A of X, we define the following operator: $A^{p^*}(I, \tau) = \{x \in X{:}\ A\cap U\notin I$ for every $U \in\tau^p(x)\}$. In case there is no confusion $A^{p^*}(I, \tau)$ is briefly denoted by A^{p^*} and is called the pre-local function of A with respect to I and τ.

Lemma.3.1. Let (X, τ, I) be an ideal topological space. Then $A^{p^*}(I, \tau) \subseteq A^*(I, \tau)$ for every subset A of X.

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Proof. Let $x \in A^{p^*}(I, \tau)$. Then, $A \cap U \notin I$ for every $U \in \tau^p(x)$. Since every open set is pre-open, therefore $x \in A^*(I, \tau)$. Converse is not true in general, it is shown in Example.3.1.

Example.3.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, and $I=\{\emptyset, \{a\}\}$. Let $A = \{b, c\}$. Then $A^*(I, \tau) = \{a, b, c\} = Cl(A^*)$ and $A^{p^*}(I, \tau) = \{b, c\} = PCl(A^{p^*})$. Then $A^{p^*}(I, \tau) \subseteq A^*(I, \tau)$.

Lemma.3.2. Let (X, τ, I) be an ideal topological space and A subset of X. Then;
1) $A^{p^*}(I, \tau) = A^*(I, \tau)$ if $\tau^p = \tau$,
2) The simplest ideals are $\{\emptyset\}$ and P (X). We observe that $A^{p^*}(\{\emptyset\}) = PCl(A) \neq Cl(A)$ and $A^{p^*}(P(X)) = \emptyset$ gives $A^{p^*}(P(X)) = \emptyset$ for every $A \subseteq X$,
3) Neither $A \subseteq A^{p^*}$ nor $A^{p^*} \subseteq A$ in general.

Example.3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ with $I = \{\emptyset\}$. If $A = \{b, c\}$, then $A^* = \{a, b, c\} = Cl(A) = Cl(A^*)$ and $A^{p^*}=\{b, c\} = PCl(A) = PCl(A^{p^*}) \neq Cl(A)$. Again for $I = P(X)$ and for any subset A of X, $A^* = \emptyset$ gives $A^{p^*} = \emptyset$.

Theorem.3.1. Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then, for pre-local functions, the following properties hold:
1) $\emptyset^{p^*} = \emptyset$,
2) If $A \subseteq B$, then $A^{p^*} \subseteq B^{p^*}$,
3) If I_1, I_2 tow ideal on X, $I_1 \subseteq I_2$, then $A^{p^*}(I_2) \subseteq A^{p^*}(I_1)$,
4) $A^{p^*} = PCl(A^{p^*}) \subseteq PCl(A)$ and A^{p^*} is pre-closed in(X, τ)
5) $(A^{p^*})^{p^*} \subseteq A^{p^*}$,
6) $(A \cup B)^{p^*} = A^{p^*} \cup B^{p^*}$,
7) $(A \cap B)^{p^*} \subseteq A^{p^*} \cap B^{p^*}$,
8) $A^{p^*} - B^{p^*} = (A - B)^{p^*} - B^{p^*} \subseteq (A - B)^{p^*}$,
9) If $U \in \tau$, then $U \cap A^{p^*} = U \cap (A \cap U)^{p^*} \subseteq (A \cap U)^{p^*}$,
10) If $E \in I$, then $E^{p^*} = \emptyset$,
11) If $E \in I$, then $(A \cup E)^{p^*} = A^{p^*} = (A - E)^{p^*}$.

Proof. 1) The proof is obvious.

2) Let $A \subseteq B$ and $x \notin B^{p^*}$. Then there exists $U \in \tau^p(x)$ such that $U \cap B \in I$. Since $A \subseteq B$, $U \cap A \in I$ and $x \notin A^{p^*}$. This proves that $A^{p^*} \subseteq B^{p^*}$.

3) Let that $x \notin A^{p^*}(I_1)$. Then there exists $U \in \tau^p(x)$ such that $A \cap U \in I_1$. Since $I_1 \subseteq I_2$, $A \cap U \in I_2$ and $x \notin A^{p^*}(I_2)$. Therefore, $A^{p^*}(I_2) \subseteq A^{p^*}(I_1)$.

4) We have $A^{p^*} \subseteq PCl(A^{p^*})$ in general. Let $x \in PCl(A^{p^*})$. Then $A^{p^*} \cap U \neq \emptyset$ for every $U \in \tau^p(x)$. Then therefore, there exists some $y \in A^{p^*} \cap U$ and $U \in \tau^p(y)$. Since $y \in A^{p^*}$, $A \cap U \notin I$ and $x \in A^{p^*}$. Then we have $PCl(A^{p^*}) \subseteq A^{p^*}$. Again, let $x \in PCl(A^{p^*}) = A^{p^*}$, then $U \cap A \notin I$ for every $U \in \tau^p(x)$. This implies $U \cap A \neq \emptyset$ for every $U \in \tau^p(x)$. Therefore, $x \in PCl(A)$. This proves $A^{p^*} = PCl(A^{p^*}) \subseteq PCl(A)$.

5) Let $x \in (A^{p^*})^{p^*}$. Then, $U \cap A^{p^*} \notin I$ for every $U \in \tau^p(x)$, and hence $U \cap A^{p^*} \neq \emptyset$. Let $y \in U \cap A^{p^*}$. Then $U \in \tau^p(y)$ and $y \in A^{p^*}$. Hence we have $U \cap A \notin I$ and $x \in A^{p^*}$. Hence $(A^{p^*})^{p^*} \subseteq A^{p^*}$.

6) We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then from (2), $A^{p^*} \subseteq (A \cup B)^{p^*}$ and $B^{p^*} \subseteq (A \cup B)^{p^*}$. Hence $A^{p^*} \cup B^{p^*} \subseteq (A \cup B)^{p^*}$. Let $x \in (A \cup B)^{p^*}$. Then, $(U \cap A) \cup (U \cap B) = U \cap (A \cup B) \notin I$. Therefore, $U \cap A \notin I$ or $U \cap B \notin I$ for every $U \in \tau^p(x)$. This implies that $x \in A^{p^*}$ or $x \in B^{p^*}$, that is $x \in A^{p^*} \cup B^{p^*}$. Therefore, we have $(A \cup B)^{p^*} \subseteq A^{p^*} \cup B^{p^*}$. Hence, we obtain $(A \cup B)^{p^*} = A^{p^*} \cup B^{p^*}$.

7) We have $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$. Then from (2), $(A \cap B)^{p^*} \subseteq A^{p^*}$ and $(A \cap B)^{p^*} \subseteq B^{p^*}$.

Hence $(A \cap B)^{p^*} \subseteq A^{p^*} \cap B^{p^*}$.

8) Since $A = (A-B) \cup (B \cap A)$, by (6) we have $A^{p^*} = (A - B)^{p^*} \cup (A \cap B)^{p^*} \subseteq (A - B)^{p^*} \cup A^{p^*}$.
Thus $A^{p^*} - B^{p^*} \subseteq (A - B)^{p^*} - B^{p^*}$.

Now $(A - B)^{p^*} \subseteq A^{p^*}$ by (2), $(A - B)^{p^*} - B^{p^*} \subseteq A^{p^*} - B^{p^*}$. Hence $A^{p^*} - B^{p^*} = (A - B)^{p^*} - B^{p^*} \subseteq (A - B)^{p^*}$.

9) Let $U \in \tau$ and $x \in U \cap A^{p^*}$. Then, $x \in U$ and $x \in A^{p^*}$. Since $U \in \tau$ then $V \in \tau^p(x)$ such that $x \in V$. Then $V \cap U \in \tau^p(x)$ and $V \cap (U \cap A) = (V \cap U) \cap A \notin I$. Then $x \in (A \cap U)^{p^*}$ and hence we obtain $U \cap A^{p^*} \subseteq (A \cap U)^{p^*}$. Moreover, $U \cap A^{p^*} \subseteq U \cap (A \cap U)^{p^*}$, by(2) $(A \cap U)^{p^*} \subseteq A^{p^*}$ and $U \cap (A \cap U)^{p^*} \subseteq U \cap A^{p^*}$. Therefore, $U \cap A^{p^*} = U \cap (A \cap U)^{p^*} \subseteq (A \cap U)^{p^*}$.

10) Let $x \in E^{p^*}$. Then for any $U \in \tau^p(x)$, $E \cap U \notin I$.

But since $E \in I$, $E \cap U \in I$ for every $U \in \tau^p(x)$. This is a contradiction. Hence $E^{p^*} = \emptyset$.

11) Since $E \in I$, by (10) $E^{p^*} = \emptyset$. By (8), $A^{p^*} = (A - E)^{p^*}$ and by (6) $(A \cup E)^{p^*} = A^{p^*} \cup E^{p^*} = A^{p^*}$. Hence we obtain $(A \cup E)^{p^*} = A^{p^*} = (A - E)^{p^*}$.

Theorem.3.2. Let (X, τ) be a topological space with ideals I_1 and I_2 on X and A subset of X. Then, $A^{p^*}(I_1 \cap I_2) = A^{p^*}(I_1) \cup A^{p^*}(I_2)$.

Proof. By Theorem.3.1.(3) we have $A^{p^*}(I_1) \subseteq A^{p^*}(I_1 \cap I_2)$ and $A^{p^*}(I_2) \subseteq A^{p^*}(I_1 \cap I_2)$. Therefore, we obtain $A^{p^*}(I_1) \cup A^{p^*}(I_2) \subseteq A^{p^*}(I_1 \cap I_2)$. Now, let $x \in A^{p^*}(I_1 \cap I_2)$. Then, for every $U \in \tau^p(x)$, $U \cap A \notin I_1 \cap I_2$ and hence $U \cap A \notin I_1$ or $U \cap A \notin I_2$. This shows that $x \in A^{p^*}(I_1)$ or $x \in A^{p^*}(I_2)$. Therefore, we have $x \in A^{p^*}(I_1) \cup A^{p^*}(I_2)$. This shows that $A^{p^*}(I_1 \cap I_2)$

$\subseteq A^{p^*}(I_1) \cup A^{p^*}(I_2)$. Then, we obtain $A^{p^*}(I_1 \cap I_2) = A^{p^*}(I_1) \cup A^{p^*}(I_2)$.

Definition.3.2. Let (X, τ, I) be an ideal topological space and A subset of X. Then $PCl^*(A) = A \cup A^{p^*}$ is a calledclosure operator.

Theorem.3.3. Let (X, τ, I) be an ideal topological space and A, B subset of X. Then;

1) $PCl^*(\emptyset) = \emptyset$,
2) $A \subseteq PCl^*(A)$,
3) $PCl^*(A \cup B) = PCl^*(A) \cup PCl^*(B)$,
4) $PCl^*(A) = PCl^*(PCl^*(A))$.

Proof.1) $PCl^*(\emptyset) = \phi^{p^*} \cup \emptyset$, by Theorem.3.1.(2) then $PCl^*(\emptyset) = \emptyset$
2) $A \subseteq A \cup A^{p^*} = PCl^*(A)$.

3) $PCl^*(A \cup B) = (A \cup B) \cup (A \cup B)^{p^*} = (A \cup B) \cup (A^{p^*} \cup B^{p^*})$ by Theorem.3.1(6).
Hence $PCl^*(A \cup B) = (A \cup B) \cup (A^{p^*} \cup B^{p^*}) = (A^{p^*} \cup A) \cup (B^{p^*} \cup B) = PCl^*(A) \cup PCl^*(B)$.

4) $PCl^*(PCl^*(A)) = PCl^*(A \cup A^{p^*}) = (A \cup A^{p^*})^{p^*} \cup (A \cup A^{p^*}) = ((A^{p^*})^{p^*} \cup A^{p^*}) \cup (A \cup A^{p^*}) = A \cup A^{p^*} = PCl^*(A)$.

Theorem.3.4. Let (X, τ, I) be an ideal topological space and A, B subset of X. Then

1) If $A \subseteq B$, then $PCl^*(A) \subseteq PCl^*(B)$,
2) $PCl^*(A \cap B) \subseteq PCl^*(A) \cap PCl^*(B)$.

Proof. This is obvious by Theorem.3.1.

Theorem.3.5. Let (X, τ, I) be an ideal topological space. Then $\tau^{p^*}(I) = \{A \in P(X) : PCl^*(A^c) = A^c\}$ is topology on X and finer that τ^p .When there is no ambiguity we will writeτ^{p^*} for $\tau^{p^*}(I)$.

Proof. This is obvious by Theorem.3.1, and Theorem.3.3. Again byTheorem.3.1,(4), we have $A^{p^*} \subseteq PCl(A)$, then $A \cup A^{p^*} \subseteq A \cup PCl(A) = PCl(A)$, then $PCl^*(A) \subseteq PCl(A)$. Hence $\tau^p \subseteq \tau^{p^*}(I)$.

Example.3.3. Let (X, τ, I) be an ideal topological space and A subset of X. If $I = \{\emptyset\}$, then$\tau^p = \tau^{p^*}(I)$. In fact, if $x \in PCl(A)$, then, $U \cap A \neq \emptyset$ for every $U \in \tau^p(x)$ then $U \cap A \notin \{\emptyset\} = I$ then $x \in A^{p^*}$. Hencex $\in A \cup A^{p^*} = PCl^*(A)$ then $PCl(A) \subseteq PCl^*(A)$ but by Theorem.3.5.$PCl^*(A) \subseteq PCl(A)$. Hence $PCl^*(A) = PCl(A)$. Consequently, $\tau^p = \tau^{p^*}(\emptyset)$.

Theorem.3.6. Let (X, τ) be a topological space and let I_1, I_2 be two ideals on X. Then
If $I_1 \subseteq I_2$, then $\tau^{p^*}(I_1) \subseteq \tau^{p^*}(I_2)$.

Proof. Straight forward.

Theorem.3.7. Let (X, τ, I) be an ideal topological space. Then $\beta(I, \tau) = \{U - E : U \in \tau^p(x), E \in I\}$ is a basis for τ^{p^*}.

Proof. Since $\emptyset \in I$, then $U - \emptyset = U \in \tau^p(x)$ and $\tau^p(x) \subseteq \beta$ from which it follows that $X = \cup\beta$ (recall that pre-open sets is forms a topology). Also $\beta_1, \beta_2 \in \beta$, and I_1; $I_2 \in I$, we have $\beta_1 = U_1 - I_1$ and $\beta_2 = U_2 - I_2$, where $U_1, U_2 \in \tau^p(x)$. Then $\beta_1 \cap \beta_2 = (U_1 - I_1) \cap (U_2 - I_2) = (U_1 \cap (X - I_1)) \cap (U_2 \cap (X - I_2)) = (U_1 \cap U_2) - (I_1 \cup I_2) \in \beta$, where $U_1 \cap U_2 \in \tau^p(x)$, $I_1 \cup I_2 \in I$.

4. Pre-compatibility of topological spaces

Definition.4.1. [3] Let (X, τ, I) be an ideal topological space. We say the topology $\tau$is compatible with the ideal I, denoted $\tau \sim I$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists $U \in N(x)$ such that $U \cap A \in I$, then $A \in I$, where $N(x)$denotes the open neighborhood system at x.

Definition.4.2. Let (X, τ, I) be an ideal topological space. We say the topology τ is pre-compatible with the ideal I, denoted $\tau \eqsim I$, if the following holds for every $A \subseteq X$:if for every $x \in A$ there exists $U \in \tau^p(x)$ such that $U \cap A \in I$, then $A \in I$.

Remark.4.1. A compatible space is pre-compatible but the converse is not true in general.

Theorem.4.1. Let (X, τ, I) be an ideal topological space, the following properties are equivalent;

1) $\tau \eqsim I$,
2)If a subset A of X has a cover of pre-open sets each of whose intersection with A is in I, then$A \in I$,
3) For every $A \subseteq X$, $A \cap A^{p^*} = \emptyset$ implies that $A \in I$,4) For every $A \subseteq X$, $A - A^{p^*} \in I$,
5) For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^{p^*}$,then $A \in I$.

Proof.1) $\Rightarrow$2) The proof is obvious.
2) $\Rightarrow$ 3) Let $A \subseteq X$ and $x \in A$. Then $x \notin A^{p^*}$ and there exists $U \in \tau^p(x)$ such that $U \cap A \in I$. Therefore, we have $A \subseteq \cup\{U : x \in A\}$ and $U \in \tau^p(x)$ and by (2) $A \in I$.
3) $\Rightarrow$ 4) For any $A \subseteq X$, $A - A^{p^*} \subseteq A$ and $(A - A^{p^*}) \cap (A - A^{p^*})^{p^*} \subseteq (A - A^{p^*}) \cap A^{p^*} = \emptyset$. By (3), $A - A^{p^*} \in I$.
4) $\Rightarrow$ 5) By (4), for every $A \subseteq X$, $A - A^{p^*} \in I$. Let $A - A^{p^*} = E \in I$, then $A = E \cup (A \cap A^{p^*})$ and by Theorem.3.1(6) $A^{p^*} = E^{p^*} \cup (A \cap A^{p^*})^{p^*} = (A \cap A^{p^*})^{p^*}$ because Theorem.3.1(10). Therefore, we have $A \cap A^{p^*} = A \cap (A \cap A^{p^*})^{p^*} \subseteq (A \cap A^{p^*})^{p^*}$and $A \cap A^{p^*} \subseteq A$. By the assumption $A \cap A^{p^*} = \emptyset$ and hence $A = A - A^{p^*} \in I$.

5) $\Rightarrow$ 1) Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in \tau^p(x)$ such that $U \cap A \in I$. Then

$A\cap A^{p^*} = \emptyset$. Since $(A - A^{p^*}) \cap (A - A^{p^*})^{p^*} \subseteq (A - A^{p^*}) \cap A^{p^*} = \emptyset$, $A - A^{p^*}$ contains no nonempty subset B with $B \subseteq B^{p^*}$. By (5), $A - A^{p^*} \in I$ and hence $A = A\cap(X-A^{p^*}) = A - A^{p^*} \in I$.

Theorem.4.2. Let (X, τ, *I*) be an ideal topological space. If τ is pre-compatible with ideal *I*, then the following properties are equivalent;
1) For every $A\subseteq X$, $A\cap A^{p^*} = \emptyset$ implies that $A^{p^*} = \emptyset$,
2) For every $A\subseteq X$, $(A - A^{p^*})^{p^*} = \emptyset$,
3) For every $A\subseteq X$, $(A \cap A^{p^*})^{p^*} = A^{p^*}$.

Proof. First, we show that (1) holds if τ is pre-compatible with ideal *I*. Let A be any subset of X and $A\cap A^{p^*} = \emptyset$. By Theorem 4.1(3) $A\in I$ and by Theorem.3.1(10)$A^{p^*} = \emptyset$.
1) ⇒2) Assume that for every $A\subseteq X$, $A\cap A^{p^*} = \emptyset$ implies that$A^{p^*} = \emptyset$. Let $B = A - A^{p^*}$, then $B\cap B^{p^*} = (A - A^{p^*}) \cap (A - A^{p^*})^{p^*} = (A\cap (X - A^{p^*})) \cap (A\cap(X - A^{p^*}))^{p^*} \subseteq (A\cap (X - A^{p^*}))\cap(A^{p^*}\cap(X - A^{p^*})^{p^*}) = \emptyset$.By (1), we have $B^{p^*} = \emptyset$. Hence $(A - A^{p^*})^{p^*} = \emptyset$.
2) ⇒3) Assume for every $A\subseteq X$, $(A - A^{p^*})^{p^*} = \emptyset$.$A = (A - A^{p^*})\cup (A \cap A^{p^*})$, then $A^{p^*} = [(A - A^{p^*}) \cup (A \cap A^{p^*})]^{p^*} = (A - A^{p^*})^{p^*} \cup (A \cap A^{p^*})^{p^*} = (A \cap A^{p^*})^{p^*}$.
3) ⇒1) Assume for every $A\subseteq X$, $A \cap A^{p^*} = \emptyset$ and $(A \cap A^{p^*})^{p^*} = A^{p^*}$.This implies that$A^{p^*} = \emptyset$.

Theorem.4.3. Let (X, τ, *I*) be an ideal topological space, then the following properties are equivalent;
1) $\tau^p \cap I = \emptyset$,
2) If $E\in I$, then $PInt(E) = \emptyset$,
3) For every $V\in\tau^p$, $V\subseteq V^{p^*}$,
4) $X = X^{p^*}$.

Proof.1) ⇒2) Let $\tau^p \cap I = \emptyset$ and $E\in I$. Suppose that $x \in PInt(E)$.Then there exists $U\in\tau^p$ such that $x \in U \subseteq E$. Since $E\in I$ and hence $\emptyset \neq \{x\}\subseteq U \in \tau^p \cap I$. This is contrary that $\tau^p \cap I = \emptyset$. Therefore, $P\,Int(E) = \emptyset$. 2) ⇒3) Let $x \in V$. Assume $x \notin V^{p^*}$ then there exists $U\in\tau^p(x)$ suchthat $V\cap U \in I$. By (2), $x \in V\cap U = PInt(V\cap U) = \emptyset$. Hence, $x \in V^{p^*}$and $V\subseteq V^{p^*}$. 3) ⇒4) Since X is pre-open, then $X = X^{p^*}$. 4) ⇒1) $X = X^{p^*} = \{x \in X : U \cap X = U \notin I$ for every $U\in\tau^p(x)\}$. Hence $\tau^p \cap I = \emptyset$.

5. Pre-open set operator ψ

Definition.5.1. Let (X, τ, *I*) be an ideal topological space. An operator $\psi:P(X) \rightarrow \tau$ is defined as follows; for every $A \in X$, $\psi(A)=\{x\in X$: there exists $U\in\tau^p(x)$ such that $U-A \in I\}$.We observe that $\psi(A) = X -(X - A)^{p^*}$. The behavior of the operator ψ has been discussed in the following theorem.

Theorem.5.1. Let (X, τ, *I*) be an ideal topological space. Let A, B subset set of X. Then;

1) ψ(A) is pre-open set,
2)$PInt(A)\subseteq\psi(A)$,
3) If $A\subseteq B$, then $\psi(A) \subseteq \psi(B)$,
4)$\psi(A\cap B)=\psi(A)\cap\psi(B)$,
5)$\psi(A\cup B)=\psi(A)\cup\psi(B)$,
6) If $U\in \tau^p$, then $U \subseteq \psi(U)$,
7) $\psi(A)\subseteq\psi((\psi(A))$,
8) $\psi(A) = \psi(\psi(A))$
if and only if $(X - A)^{p^*} = ((X - A)^{p^*})^{p^*}$,
9)If $(A - B)\cup(B - A) \in I$, then $\psi(A) = \psi(B)$,
10) If $E\in I$, then $\psi(E) = X - X^{p^*}$,
11) If $E\in I$, then $\psi(A-E) = \psi(A)$,
12) If $E\in I$, then $\psi(A\cup E) = \psi(A)$.

Proof.1) This follows from Theorem.3.1 (2).
2) From definition of ψ operator, $\psi(A) = X-(X - A)^{p^*}$. Then $X-PCl(X -A)\subseteq X- (X - A)^{p^*}=\psi(A)$. Hence $PInt(A)\subseteq\psi(A)$.
3)Let $A\subseteq B$, then $(X - B)\subseteq(X - A)$. Then from Theorem.3.1, (4),$(X - B)^{p^*}\subseteq(X - A)^{p^*}$then $\psi(A) \subseteq\psi(B)$.
4) $\psi(A\cap B) = X - (X - (A \cap B))^{p^*} = X - ((X - A) \cup (X - B))^{p^*} = X - ((X - A)^{p^*} \cup (X - B)^{p^*})^{p^*} = [X - (X - A)^{p^*}] \cap [X - (X - B)^{p^*}] = \psi(A) \cap \psi(B)$.5) $\psi(A\cup B) = X - (X - (A \cup B))^{p^*} = X - ((X - A) \cap (X - B))^{p^*} = X - ((X - A)^{p^*} \cap (X - B)^{p^*})^{p^*} = [X - (X - A)^{p^*}] \cup [X - (X - B)^{p^*}] = \psi(A) \cup\psi(B)$.
6) Let $U\in \tau^p$. Then (X – U) is a pre-closed set and hence $PCl(X - U) = (X - U)$.
Then$(X - U)^{p^*}\subseteq PCl(X - U) = (X - U)$. Hence $U\subseteq X-(X - U)^{p^*}$, so$U\subseteq\psi(U)$.
7) From (1), $\psi(A) \in \tau^p$, and from (6), $\psi(A) \subseteq\psi(\psi(A))$.
8) This follows from the facts:
1. $\psi(A)= X -(X - A)^{p^*}$.
2. $\psi(\psi(A)) = X-[X - (X - (X - A)^{p^*})]^{p^*} = X-((X - A)^{p^*})^{p^*}$.
9) Let $(A - B)\cup(B - A) \in I$, and let $A - B = E_1$, $B - A = E_2$. We observe that $E_1,E_2\in I$ by heredity, and $B = (A - E_1)\cup E_2$. Thus $\psi(A) = \psi(A - E_1) = \psi((A - E_1)\cup E_2) = \psi(B)$.
10) By Theorem.3.1,(10) we obtain if $E\in I$, then $\psi(E) = X- X^{p^*}$.
11) This follows from Theorem.3.1,(10)and $\psi(A - E) = X -[X - (A - E)]^{p^*} = X -[(X - A) \cup E]^{p^*} = X - (X - A)^{p^*} = \psi(A)$.
12) This follows from Theorem.3.1,(9)and $\psi(A\cup E) = X -[X - (A \cup E)]^{p^*} = X -[(X - A) - E]^{p^*} = X -(X - A)^{p^*} = \psi(A)$.

Theorem.5.2. Let (X, τ, I) be an ideal topological space. If $\eta = \{A \subseteq X : A \subseteq \psi(A)\}$. Then η is a topology for X.

Proof. Let $\eta = \{A \subseteq X : A \subseteq \psi(A)\}$. By Theorem.3.1,(1)$\phi^{p^*} = \emptyset$ and $\psi(X) = X - (X - X)^{p^*} = X - \phi^{p^*} = X$. Moreover, $\psi(\emptyset) = X - (X - \emptyset)^{p^*} = X - X = \emptyset$. Therefore, we obtain that $\emptyset \subseteq \psi(\emptyset)$ and $X \subseteq \psi(X) = X$, and thus $\emptyset$ and $X \in \eta$. Now if $A, B \in \eta$, then by Theorem.5.1 $A \cap B \subseteq \psi(A) \cap \psi(B) = \psi(A \cap B)$ which implies that $A \cap B \in \eta$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \eta$, then $A_\alpha \subseteq \psi(A_\alpha) \subseteq \psi(\cup A_\alpha)$ for every α and hence $\cup A_\alpha \subseteq \psi(\cup A_\alpha)$. This shows that η is a topology.

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